Math 251
Exam 3 part 1
Spring 2016

Name:____________________

Show All Work

1) Use Lagrange Multipliers to find the maximum and minimum values of the function subject to the given constraint.

$$f(x,y,\bullet) = x^2 + y^2;\ \ where\ xy = 1$$

$\nabla f = \langle 2x, 2y \rangle \qquad \nabla g = \langle y, x \rangle$

$\left.\begin{matrix} \nabla f = \lambda \nabla g \\ xy = 1 \end{matrix}\right\rangle \quad \left.\begin{matrix} 2x = \lambda y \\ 2y = \lambda x \\ xy = 1 \end{matrix}\right\} \longrightarrow \begin{matrix} \lambda = \frac{2x}{y} \text{ or } y = 0 \\ \lambda = \frac{2y}{x} \text{ or } x = 0 \\ xy = 1 \ (\text{so } x \neq 0,\ y \neq 0) \end{matrix}$

So we have $\frac{2x}{y} = \frac{2y}{x}$ and $xy = 1$

$2x^2 = 2y^2$

$x^2 = y^2$

$x = \pm y$ and $xy = 1$

so $x^2 = 1$

$x = \pm 1$ points $(1,1)$ and $(-1,-1)$

$f(1,1) = 2$

$f(-1,-1) = 2$

which is it? MAX or min?

two methods

1) check any other point satisfying $xy = 1$

$f(2, \frac{1}{2}) = 2^2 + \frac{1}{2}^2 = 4 + \frac{1}{4} > 2$

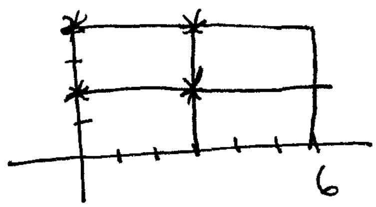
"2" must be a minimum

$\boxed{\text{min value } 2}$

2) NOTE $z = x^2 + y^2$ is in general unbounded above since $xy = 1$ allows for points such as $(100, \frac{1}{100})$ $(1{,}000{,}000, \frac{1}{1{,}000{,}000})$ we see that even with the constraint $z = x^2 + y^2$ is unbounded above

Math 251 name Key
Exam III part 2
Spring 2016 Show All Work

1) If R = [0, 6] x [0, 4], use a Riemann sum with m = n = 2 to estimate the value of $\iint_R \sqrt{x+xy+y}\, dA$ Take the sample points to be the upper left corners of the rectangles.



$\Delta A = \Delta x \cdot \Delta y = 3 \cdot 2 = 6$

$$\iint_R \sqrt{x+xy+y}\, dA \approx 6 \cdot \left[f(0,2) + f(3,2) + f(0,4) + f(3,4) \right]$$

$$= 6 \left[\sqrt{2} + \sqrt{11} + \sqrt{4} + \sqrt{19} \right]$$

$$\approx 66.5$$

2) Calculate the following:

a) $\int_0^{\pi} \int_{\pi-y}^{\pi+y} \sin(x+y)\, dxdy$

$$= \int_0^{\pi} -\cos(x+y) \Big|_{x=\pi-y}^{x=\pi+y} dy = \int_0^{\pi} \left(-\cos(\pi+2y) + \cos\pi\right) dy$$

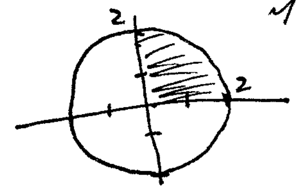
$$= \int_0^{\pi} \left(-\cos(\pi+2y) - 1\right) dy = -\frac{\sin(\pi+2y)}{2} - y \Big|_0^{\pi}$$

$$= \left[-\frac{\sin(3\pi)}{2} - \pi\right] - \left[-\frac{\sin\pi}{2} + 0\right]$$

$$= (0-\pi) - (0+0) = -\pi$$

3) Find the volume of the solid bounded by $x^2 + y^2 = 4$, $z = x + y$ in the first octant

a) Using rectangular coordinates



if $z = 0$

$x+y=0$

$y=-x$

does not cross in 1st octant

$x^2+y^2=4$

$y^2=4-x^2$

$y=\pm\sqrt{4-x^2}$

in first quadrant $y=\sqrt{4-x^2}$

$$\int_0^2\int_0^{\sqrt{4-x^2}}(x+y)\,dy\,dx = \int_0^2 xy+\frac{y^2}{2}\Big|_0^{\sqrt{4-x^2}} dx = \int_0^2\left(x\sqrt{4-x^2}+\frac{4-x^2}{2}\right)dx$$

$$=\int_0^2 x\sqrt{4-x^2}\,dx+\int_0^2 2-\frac{x^2}{2}\,dx$$

1st integral

$u=4-x^2$

$du=-2x\,dx$

$\frac{du}{-2}=x\,dx$

$$\int_4^0 \sqrt{u}\cdot\frac{du}{-2} + \left(2x-\frac{x^3}{6}\right)\Big|_0^2 = -\frac{1}{2}\,\frac{u^{3/2}}{3/2}\Big|_4^0 + \left(4-\frac{8}{6}\right)-(0-0)$$

$$= -\frac13(0-8)+\left(4-\frac43\right)=\frac83+4-\frac43$$

$$=\frac{16}{3}$$

b) Using polar coordinates

$$\int_0^{\pi/2}\int_0^2(r\cos\theta+r\sin\theta)\,r\,dr\,d\theta$$

$$=\int_0^{\pi/2}\int_0^2 r^2(\cos\theta+\sin\theta)\,dr\,d\theta$$

$$=\left(\int_0^{\pi/2}(\cos\theta+\sin\theta)\,d\theta\right)\cdot\left(\int_0^2 r^2\,dr\right)$$

$$=\left(\sin\theta-\cos\theta\Big|_0^{\pi/2}\right)\left(\frac{r^3}{3}\Big|_0^2\right)=\big((1-0)-(0-1)\big)\left(\frac83-0\right)$$

$$=2\cdot\frac{8}{3}=\frac{16}{3}$$

4) Evaluate the following integral by switching to polar coordinates.

$\iint_R \sqrt{x^2+y^2}\, dA$ Where R is the region between the circles with center at the origin and radii 1 and 4.

$$\int_0^{2\pi}\int_1^4 \sqrt{r^2}\cdot r\, dr\, d\theta$$ note $r > 0$ so $\sqrt{r^2} = r$

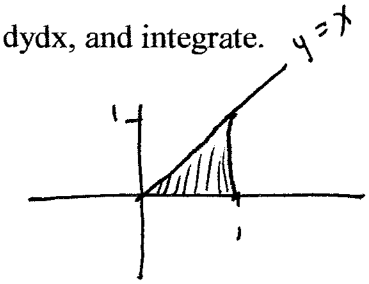
$$= \int_0^{2\pi}\int_1^4 r\cdot r\, dr\, d\theta = \int_0^{2\pi}\int_1^4 r^2\, dr\, d\theta = \int_0^{2\pi} \frac{r^3}{3}\Big|_1^4\, d\theta$$

$$= \int_0^{2\pi}\left(\frac{64}{3}-\frac{1}{3}\right)d\theta = \int_0^{2\pi} 21\, d\theta = 21\theta\Big|_0^{2\pi} = 42\pi$$

5) Sketch the region of integration, change the order of integration to dydx, and integrate.

$$\int_0^1\int_y^1 \frac{\sin x}{x}\, dx\, dy$$

y=x



$$\int_0^1\int_0^x \frac{\sin x}{x}\, dy\, dx = \int_0^1 \frac{\sin x}{x}\cdot y\Big|_0^x\, dx$$

$$= \int_0^1 \frac{\sin x}{x}\cdot x\, dx = \int_0^1 \sin x\, dx = -\cos x\Big|_0^1$$

$$= -\cos 1 + 1 = \underline{1-\cos 1}$$

6) Evaluate

$$\int_0^3\int_z^{z+2}\int_y^{y+z} 2x\,dx\,dy\,dz$$

$$\int_0^3\int_z^{z+2} x^2\Big|_y^{y+z} dy\,dz = \int_0^3\int_z^{z+2} \left((y+z)^2 - y^2\right)dy\,dz$$

$$= \int_0^3\int_z^{z+2} y^2+2yz+z^2-y^2\,dy\,dz = \int_0^3\int_z^{z+2}(2yz+z^2)\,dy\,dz$$

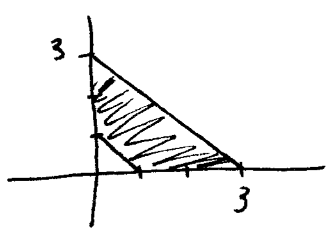
$$= \int_0^3 y^2z+z^2y\Big|_z^{z+2} dz = \int_0^3 \left[(z+2)^2\cdot z + z^2(z+2)\right] - \left[z^3+z^3\right] dz$$

$$= \int_0^3 \underline{z^3}+4z^2+4z+\underline{z^3}+2z^2\underline{-z^3-z^3}\,dz$$

$$= \int_0^3 (6z^2+4z)\,dz = 2z^3+2z^2\Big|_0^3 = (54+18)-0 = 72$$

7) **Set up** a double or triple integral for the volume of the region in the first octant that is bounded above by z = 9 - y^2 and lies between the planes x + y = 1 and x + y = 3.

$y=1-x$ $\qquad$ $y=3-x$



$$= \int_0^1\int_{1-x}^{3-x}(9-y^2)\,dy\,dx + \int_1^3\int_0^{3-x}(9-y^2)\,dy\,dx$$